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Modeling Heterogeneous Network Interference Using Poisson Point Processes

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ABSTRACT—*Cellular systems are becoming more heterogeneous with the introduction of low power nodes including femtocells, re- lays, and distributed antennas. Unfortunately, the resulting inter- ference environment is also becoming more complicated, making evaluation of different communication strategies challenging in both analysis and simulation. Leveraging recent applications of stochastic geometry to analyze cellular systems, this paper pro- poses to analyze downlink performance in a fixed-size cell, which is inscribed within a weighted Voronoi cell in a Poisson field of interferers. A nearest out-of-cell interferer, out-of-cell interferers outside a guard region, and cross-tier interferers are included in the interference calculations. Bounding the interference power as a function of distance from the cell center, the total interference is characterized through its Laplace transform. An equivalent marked process is proposed for the out-of-cell interference under additional assumptions. To facilitate simplified calculations, the interference distribution is approximated using the Gamma distribution with second order moment matching. The Gamma approximation simplifies calculation of the success probability and average rate, incorporates small-scale and large-scale fading, and works with co-tier and cross-tier interference. Simulations show that the proposed model provides a flexible way to characterize outage probability and rate as a function of the distance to the cell edge.*
Index Terms—*Wireless communication, cellular networks, MIMO, stochastic geometry, interference.*

I. INTRODUCTION

CELLULAR network deployment is taking on a massively heterogeneous character as a variety of infrastructure isbeing deployed including macro, pico, and femto base stations [1], as well as fixed relay stations [2] and distributed antennas [3]. A major challenge in deploying heterogeneous cellular net- works is managing interference. The problem is compounded by the observation that the deployment of much of the small cell infrastructure will be demand based and more irregular than tra- ditional infrastructure deployments [4]. Further, as urban areas are built out, even macro and micro base station infrastructure locations are becoming less like points on a hexagonal lattice and more random [5]. Consequently, the aggregate interference environment is more complicated to model, and evaluating the performance of communication techniques in the presence of heterogeneous interference is challenging.

Stochastic geometry can be used to develop a mathematical framework for dealing with interference [6]–[9]. With sto- chastic geometry, interferer locations are distributed according to a point process, often the homogeneous Poisson Point Process (PPP) for its tractability. PPPs have been used to model co-channel interference from macro cellular base stations [5], [10]–[13], cross-tier interference from femtocells [4], [14], [15], co-channel interference in ad hoc networks [7]–[9], [16], and as a generic source of interference [17]–[19].

Modeling co-channel interference from other base stations as performed in [5], [10], [12], [13], [20] is a good starting point for developing insights into heterogeneous network interference. In [10] PPPs are used to model various components of a telecom- munications network including subscriber locations, base sta- tion locations, as well as network infrastructure leveraging re- sults on Voronoi tessellation. In [12], [20] a cellular system with PPP distribution of base stations, called a shotgun cellular system, is shown to lower bound a hexagonal cellular system in terms of certain performance metrics and to be a good approx- imation in the presence of shadow fading. In [5], a comprehen- sive analysis of a cellular system with a PPP is presented, where key system performance metrics like coverage probability and average rate are computed by averaging over all deployment scenarios and cell sizes. The approach in [5] is quite powerful, leading to closed form solutions for some special choices of pa- rameters. In [13] the outage and the handover probabilities in a PPP cellular network are derived for both random general slow fading and Rayleigh fading, where mobiles are attached to the base station that provides the best mean signal power. Recent work has considered extensions of [5], [12] to multi-tier net- works. For example [21] extends [12] to provide some results on the signal-to-interference ratio in multi-tier networks while [22] extends [12] to provide remarkable simple expressions for the success probability in interference limited multi-tier networks. From our perspective, the main drawbacks of the approach in [5], [10], [12], [13], [20] for application to existing systems is that performance is characterized for an entire system not for a given cell. As cellular networks are already built out, cellular providers will often want to know the performance they achieve in some given cells by adding additional infrastructure (and thus interference) in the rest of the network. It is also challenging to incorporate more complex kinds of heterogeneous infrastruc- ture like fixed relays or distributed antennas into the signal and interference models.

Models for co-channel interference from PPPs in general wireless settings have been considered in prior work [16]–[19], [23]. References [17]–[19] derive models for the complex noise distribution. These models assume that the noise changes on a sample-by-sample basis and are suitable for deriving optimum signal processing algorithms, for example optimum detectors. Other work, see e.g., [16], [23] and the references therein, provide models for the noise power distribution. Most system-level analysis work [5], [10], [12], [13], [20] focuses on the noise power distribution. They assume that the interference distribution conditioned on the noise power is Gaussian and thus characterize performance based on quantities like the signal-to-interference ratio or the signal-to-interfer- ence-plus-noise ratio, which are not appropriate for other noise distributions [17]–[19]. This is reasonable because cellular systems tend to have structured transmissions in time and frequency, hence the interfering sets are likely to be constant over a coding block. Consequently, we focus on characterizing and employing the noise power distribution in our work. We assume the full spectrum is reused among all tiers of nodes; work on the spectrum sharing can be found in [42], [45].

In this paper we propose a simplified interference model for heterogeneous networks called the hybrid approach. The key idea is to evaluate performance in a fixed-size circular cell considering co-channel interference outside a guard region and cross-tier interference from within the cell. Different sources of interference are modeled as marked Poisson point processes where the mark distribution includes a contribution from small-scale fading and large-scale fading. The fixed-size cell is inscribed inside the weighted Voronoi tessellation formed from the multiple sources of out-of-cell interference. The co-channel interference has two components: one that corresponds to the nearest interferer that is exactly twice the cell radius away (the boundary of the guard region), and one that corresponds to the other interferers (outside the guard region). An upper bound on the co-channel interference power distribution is characterized through its Laplace transform as a function of distance from the cell center for the case of downlink transmission, after lower bounding the size of the exclusion region. Performance is analyzed as a function of the "relative" distance from the receiver to the center of the cell. With suitable choice of cell radius as a function of density, the resulting analysis (in the case of a single tier of interferers) is invariant to density, making the fixed-cell a "typical" cell.

One of the key advantages of the proposed model is that no explicit user association scheme needs to be specified. User as- sociation in heterogeneous cellular networks is an important topic of investigation and prior work has considered specific user assignment schemes [22], [24]–[26]. Two metrics, namely the received signal quality and the cell traffic load, are usually considered for selecting the serving base station. In the hybrid model, users can be arbitrarily distributed in the cell of interest without requiring any assumption on the user association. More- over, our new calculations still operate under the assumption that users connect to the closest base station. In the presence of shadowing, this may not be realistic, as closest base station as- sociation does not necessarily have a geometric interpretation. In practice, however, the association may be carried out in dif- ferent granularity as compared to the channel dynamics. For in- stance, users may connect to the base station that provides the highest averaged received power (often with biasing), thus the effect of fading and shadowing becomes constant and the users connect to the closest base station on average.

To facilitate simplified calculations, we use approximations based on the Gamma distribution. We model the fading compo- nent using the Gamma distribution as this incorporates single user and multiple user signaling strategies with Rayleigh fading as special cases, along with an approximation of the composite small-scale and log-normal fading distribution [43]. We then ap- proximate the interference distribution using the Gamma distri- bution with second order moment matching; the moments are finite because we use a nonsingular path-loss model. Using the Gamma approximation, we present simplified expressions for the success probability and the average rate, assuming that the system is interference limited, i.e., our calculations are based on the signal-to-interference ratio. Simulations show that the Gamma approximation provides a good fit over most of the cell, and facilitates lower complexity.

Our mathematical approach follows that of [5] and leverages basic results on PPPs [8], [9], [23]. Compared with [5], we con- sider a fixed-size cell inscribed in a weighted Voronoi tessella- tion, and do not average over possible Voronoi cells. We also do not calculate system-wide performance measures, rather we focus on performance for a fixed cell of interest. We also demon- strate how to employ the proposed fixed cell analysis in a hetero- geneous network consisting of mixtures of the different kinds of infrastructure. The advantage of our approach is that more complex types of communication and network topologies can be analyzed in a given target cell and key insights on the perfor- mance of advanced transmission techniques with heterogeneous interference using stochastic geometry can be obtained. Com- pared with our prior work reported in [28], in this paper we make the concept of the guard region more precise in the context of random cellular networks, avoiding the ad hoc calculation of the guard radius. As a byproduct, we also modified the calculations of the interference term to include a closest interferer that sits exactly at the edge of the guard region. In addition, we deal with more general fading distributions and use the Gamma approxi- mation to simplify the calculation of the success probability and the ergodic rate. A benefit of our new model is that with a par- ticular choice of the cell radius, the results are invariant with the base station density. Guard zones have been used in other work on interference models e.g., [19], [29]. In [29] they are used to evaluate the transmission capacity in contention-based ad hoc networks not cellular networks while in [19] they are consid- ered in the derivation of the baseband noise distribution but not the noise power.

II. MATHEMATICAL PRELIMINARIES

In this paper we use of Gamma random variables to model the small-scale fading distribution, to approximate the product of the small-scale and log-normal fading distribution, and to ap- proximate the interference power. In this section, we summarize known results on Gamma random variables to make the paper more accessible.

Definition 1: A Gamma random variable with finite shape $k > 0$ and finite scale $\theta > 0$, denoted as $\Gamma[k, \theta]$ has probability distribution function

$$f_X(x) = x^{k-1}\frac{e^{-x/\theta}}{\theta^k\Gamma(k)}$$

for $x \geq 0$. The cumulative distribution function is

$$F_X(x) = \frac{\gamma(k, x\theta)}{\Gamma(k)},$$

where $\gamma(a, b) = \int_0^b t^{a-1}e^{-t}dt$ is the lower incomplete gamma function. The first two moments and the variance are

$$\mathbb{E}X = k\theta \quad \mathbb{E}X^2 = k(1+k)\theta^2 \quad \mathbb{V}X = k\theta^2.$$

The scale terminology comes from the fact that if X is $\Gamma[k, \theta]$ then with scalar $\alpha > 0$, αX is $\Gamma[k, \alpha\theta]$.

The Gamma distribution is relevant for wireless communica- tion because it includes several channel models as a special case. For example if Y is a circularly symmetric complex Gaussian with $\mathcal{N}(0,1)$ (variance 1/2 per dimension) then $X = |Y|^2$ is $\Gamma[1, 1]$, which corresponds to Rayleigh channel assumption. If X has a Chi-square distribution with $2N$ degrees of freedom, used in the analysis of diversity systems, then X is also $\Gamma[N, 2]$. If Z is Nakagami distributed [30] with parameters μ and ω then $X = Z^2$ has a $\Gamma[k, \theta]$ distribution with $k = \mu$ and $\theta = \omega/\mu$.

Consequently the Gamma distribution can be used to represent common fading distributions.

Of relevance for computing rates, the expected value of the log of a Gamma random variable has a simple form as summa- rized in the following Lemma.

Lemma 2 (Expected Log of a Gamma Random Variable): If X is $\Gamma[k, \theta]$ then $\mathbb{E}\ln X = \psi(k) + \ln(\theta)$ where $\psi(k)$ is the digamma function.

The digamma function is implemented in many numerical packages or for large x its asymptotic approximation can be used $\psi(x) = \ln(x) - 1/x + O(1/x^2)$.

The Gamma distribution is also known as the Pearson type III distribution, and is one of a family of distributions used to model the empirical distribution functions of certain data [31]. Based on the first four moments, the data can be associated with a preferred distribution. As opposed to optimizing over the choice of the distribution, we use the Gamma distribution exclu- sively because it facilitates calculations and analysis. We will approximate a given distribution with a Gamma distribution by matching the first and second order moments in what is com- monly known as moment matching [32]. The two parameters of the Gamma distribution can be found by setting the appropriate moments equal and simplifying. We summarize the result in the following Lemma.

Lemma 3 (Gamma 2nd Order Moment Match): Consider a distribution with $\mathbb{E}X$, $\mu^{(2)}$, and variance $\sigma^2 = \mu^{(2)} - \mu^2$. Then the distribution $\Gamma[k, \theta]$ with same first and second order moments has parameters

(1) $k = \frac{\mu^2}{\sigma^2}$ $\theta = \frac{\sigma^2}{\mu}$.

Note that the shape parameter is scale invariant. For example, the Gamma approximation of αX would have shape $k = \mu^2/\sigma^2$ and scale $\theta = \alpha\sigma^2/\mu$.

We will need to deal with sums of independent Gamma random variables with different parameters. There are various closed-form solutions for the resulting distribution [33], [34] (see also the references in the review article [35]). In this paper

we choose the form in [33] as it gives a distribution that is a sum of scaled Gamma distributions.

Theorem 4 (Exact Sum of Gamma Random Variables): Sup- pose that $\{Y_i\}_{i=1}^n$ are independent $\Gamma[k_i, \theta_k]$ distributed random variables. Then the probability distribution function of $Y = \sum_i Y_i$ can be expressed as

(2) $$f_Y(y) = C \sum^{\infty} \frac{c_i}{\Gamma(\rho + i)\theta_{min}^{\rho+i}} y^{\rho+i-1} e^{-y/\theta_{min}},$$

where $\theta_{min} := \min_i \theta_i$, $\rho = \sum_{i=1}^n k_i$, $C = \prod_{i=1}^n (\theta_{min}/\theta_i)^{k_i}$, and c_i is found by recursion from

(3) $$c_{i+1} = \frac{1}{i+1} \sum_{m=1}^{i+1} m\gamma_m c_{i+1-m}$$

and $\gamma_m = \sum_{i=1}^n k_i m^{-1}(1 - \theta_{min}/\theta_i)^m$.

Proof: See [33].

For practical implementation, the infinite sum is truncated; see [33] for bounds on the error. Mathematica code for com- puting the truncated distribution is found in [36]. In some cases, many terms need to be kept in the approximation to achieve an accurate result. Consequently we also pursue a moment-matched approximation of the sum distribution.

Lemma 5 (Sum of Gammas 2nd Order Moment Match): Suppose that $\{Y_i\}$ are independent Gamma distributed random variables with parameters k_i and θ_i. The Gamma distribution $\Gamma[k_y, \theta_y]$ with the same first and second order moments has

$$k_y = \frac{(\sum_i k_i\theta_i)^2}{\sum_i k_i\theta_i^2} \text{ and } \theta_y = \frac{\sum_i k_i\theta_i^2}{\sum_i k_i\theta_i}.$$

Bounds on the maximum error obtained through a moment approximation are known [32] and may be used to estimate the maximum error in the cumulative distribution functions of the Gamma approximation. In related work [37] we show that the approximation outperforms the truncated expression in (2) unless a large number of terms are kept, and therefore con- clude that the approximation is reasonable. The approximation is exact in some cases, e.g., if all the shape values are identical (in this case $\theta_i = \theta$ then $k_y = \sum_i k_i = \rho$, $\theta_y = \theta = \theta_{min}$ and $c_i = 0$).

In wireless systems, the zero-mean log-normal distribution is used to model large-scale fluctuations in the received signal power. The distribution is $f_X(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln x)^2}{2\sigma^2}}$ where σ is the shadow standard deviation, often given in dB where $\sigma_{dB} =$. Typical values of σ_{dB} are between 3 dB and 10 dB for example in 3GPP [38], [39].

Composite fading channels are composed of a contribution from both small-scale fading and large-scale fading. Consider the random variable for the product distribution HL where H is $\Gamma[k_h, \theta_h]$ and L is log-normal with parameter σ. Related work has shown that the product distribution can be reasonably mod- eled using a Gamma distribution over a reasonable range of σ_{dB} [43] by matching the first and second central moments.

Lemma 6 (Gamma Approximation of Product Distribution): Consider the random variable PHL where P is a constant, H is $\Gamma[k_h, \theta_h]$ and L is log-normal with variance σ. The Gamma

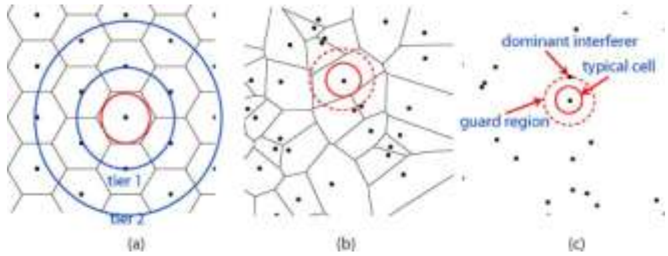


Fig. 1. Models for cellular communication. (a) Common fixed geometry model with hexagonal cells and multiple tiers of interference. (b) Stochastic geometric model where all base stations are distributed according to some 2D random process. (c) Proposed hybrid approach where there is a fixed cell of a fixed size surround by base stations distributed according to some 2D random process, with an exclusion region around the cell and a dominant interferer at the boundary of the guard region.

random variable with with the same first-order and second-order moments has

$$\Gamma[k_p, \theta_p]$$

(4) $$k_p = \frac{1}{(1/k_h + 1)e^{\sigma^2} - 1}$$

(5) $$\theta_p = (1 + k_h)\theta_h e^{3\sigma^2/2} - k_h \theta_h e^{\sigma^2/2}.$$

Because of its flexibility for modeling various kinds of small- scale fading, and composite small-scale large-scale fading, we consider general Gamma distributed fading in this paper.

III. DOWNLINK NETWORK MODEL

In the classic model for cellular systems in Fig. 1(a), base sta- tions are located at the centers of hexagons in a hexagonal tes- sellation and interference is computed in a fixed cell from mul- tiple tiers of interferers. The hexagonal model requires simula- tion of multiple tiers of interferers and makes analysis difficult. In the stochastic geometry model for cellular systems illustrated in Fig. 1(b), base stations positions follow a PPP and cells are derived from the Voronoi tessellation. However, in this model, it is difficult to study the performance at specific locations, such as the cell edge, for cell sizes are random and performance met- rics are computed in an aggregate sense accounting for all base station distributions.

Our proposed model, which we call the hybrid approach, is illustrated in Fig. 1(c). We consider a typical cell of fixed shape and size, nominally a ball with radius R_c which inscribes a Voronoi cell. A ball is assumed to simplify the analysis. The inscribing ball does miss some badly covered areas at the cell edge, thus it does not fully characterize performance in an en- tire Voronoi cell. The base station locations outside of the fixed cell are modeled according to a PPP and the interference be- comes a shot-noise process [40]. A guard region of radius R_g from the cell edge is imposed around the fixed cell in which no other transmitters can occupy. The role of the guard region is to guarantee that mobile users even at the edge of the typical cell receive the strongest signal power from the typical base station when neglecting channel fading, which also serves as the rule to systematically determine R_g. In the homogeneous network when the transmission power of all base stations is uniform, R_g R_c. In the heterogeneous case, the expression of R_g is more complicated and is discussed in Section V.

We also introduce the concept of the dominant interferer in the hybrid model, which can be better explained with one ob- servation from the stochastic model. Intuitively, we can regard the fixed cell in the hybrid approach as the inscribing ball of a Voronoi cell in the stochastic geometry model, though the size of the inscribing ball is random in the latter case. In Fig. 1(b), the size of the equivalent guard region is determined by the nearest neighboring base station, i.e., the dominant interferer locating at its outer boundary. In short, in the stochastic geometry model there always exists a dominant interferer locating at the outer boundary of the guard region. Hence, in our proposed hybrid model, besides the PPP interferers outside the guard region, we also assume one dominant interferer uniformly distributed at the edge of the guard region in the homogeneous network. In addi- tion, given R_g, the location of the dominant interferer is inde- pendent of those of all other base stations. This idea is extended to the heterogeneous case in Section V.

The choice of R_c depends on the exact application of the model. It could simply be fixed if it is desired to analyze the performance of a particular cell in a deployment. Alternatively, if it is desired to analyze a "typical" fixed cell, the radius could be chosen to be a function of the density. For example, if the cell radius corresponds to that of an equivalent PPP model of base stations with density , then $\frac{1}{4\sqrt{\lambda}}$. Then the cell radius shrinks as the network becomes more dense. Selecting R_c

=

the radius for the heterogeneous case is a bit more complex and is discussed in Section V.

Performance is evaluated within the fixed cell accounting for interference sources outside the guard region (including the dominant interferer at the edge). To make the calculations con- crete, in this paper we focus on the downlink. We consider the received signal power at distance $r = \beta R_c$ $(0 < \beta \leq 1)$ from the transmitter presumably located at the cell center. Thanks to the isotropic property of the circular cell, without loss of gener- ality, we fix the receiver location at $(r, 0)$ in the following dis- cussion. Extension to distributed antennas, e.g., requires a more complex signal model. Let the received signal power be

$$S(r) = P(r)LH,$$

where $P(r)$ $P_s/\ell(r)$ is the distant-dependent average re- ceived signal power with P_s being the transmit power, a random variable corresponding to the large-scale fading power usually H log-normal, and is a random variable corresponding to the small-scale fading power. We assume that a non-singular path-loss model is used with

$$\ell(r) = C \max(d_0, r)^n, \quad (6)$$

where is the path-loss exponent, $d_0 > 0$ is the reference distance, and $=$ is a constant. In addition to resulting in finite interference moments, (6) takes into account the realistic RF design constraints on the maximum received power and is often assumed in a standards based channel model like 3GPP LTE Advanced [38].

We consider the power due to small-scale fading as occurring after processing at the receiver, e.g., diversity combining. Fur- thermore, we assume that the transmit strategy is independent of the strategies in the interfering cells and no adaptive power

control is employed. We model the small-scale fading power as a Gamma distributed random variable with $\Gamma[k_h, \theta_h]$. The Gamma distribution allows us to model several different trans- mission techniques. For example, in conventional Rayleigh fading is while Rayleigh fading with receive an- tennas and maximum ratio combining is H $\Gamma[1, 1]$. Maximum ratio transmission with Rayleigh fading, N_r transmit antennas, and N_r receive antennas would have H $\Gamma[N_r, 1]$ is N_t.

With multiuser MIMO (MU MIMO) with transmit antennas, active users, and zero-forcing precoding then is H $\Gamma[N_t, 1]$ $=1$ where the $1/U$ follows from splitting the power among different users.

We model the large-scale fading contribution L as log-normal distributed with parameter σ^2. The large-scale fading accounts for effects of shadowing due to large objects in the environment and essentially accounts for error in the fit of the log-distance path-loss model. Because of the difficulty in dealing with com- posite distributions, we approximate the random variable with another Gamma random variable with the same first and second order moments as described in Lemma 6 with distribu- tion where k_p is found in (4) and θ_p in (5). Note that from the scaling properties of Gamma random variables, with HL this approximation, the received signal power $S(r)$ is equiva- lently $\Gamma[k_g, P(r)\theta_g]$.

IV. HOMOGENEOUS NETWORK INTERFERENCE

In this section, we develop a model for homogeneous net- work interference, i.e., interference comes from a single kind of interferer such as a macrocell. We model the interferer lo- cations according to a marked point process, where the marks correspond to the channel between the interferers and the target receiver. Specifically we consider a PPP with density and marks modeled according to the signal power distribution. For interferer k, L_k follows a log-normal distribution with while the small-scale fading distribution is . The transmit power of the interferer is denoted by . The transmit power is fixed for all interferers and the actions of each interferer are in- dependent of the other interferers. Note that the parameters of the interference mark distribution are usually different than the signal channel even when the same transmission scheme is em- ployed in all cells. For example, with single antenna transmis- sion and conventional Rayleigh fading G_k is $\Gamma[1, 1]$ for any N_r at the receiver, while with multiuser MIMO zero-forcing beam- forming with multiple transmit antennas serving $U \leq N_t$ users G_k is $\Gamma[k_g, \theta_g]$. P_g

We aim at computing the interference power received at $(r, 0)$ as illustrated in Fig. 2. In the homogeneous network, as men- tioned in the previous section, $R_g = R_c = \frac{1}{4\sqrt{\lambda}}$.

The interference I comes from two independent parts: I_0 is the interference from the dominant interferer at the guard region edge and I_1 is the interference from all other base stations which form a PPP outside the guard region, i.e.,

$$I(r) = I_0(r) + I_1(r).$$

Using the law of cosines, I_0 can be computed as

$I_0(r) =$

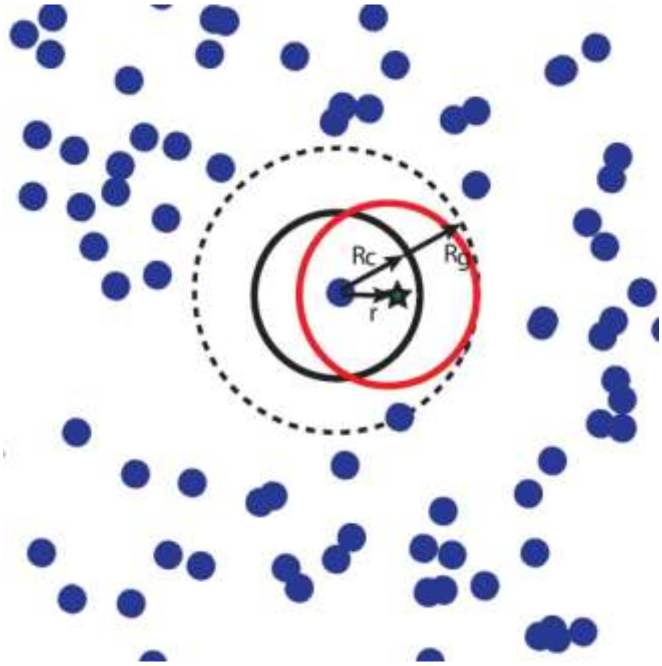


Fig. 2. Interference power calculation at a point away from the transmitter. An upper bound is considered as the interference is not excluded uniformly around the receiver.

where η is a uniform random variable on $(0, 2\pi]$, and $r = \beta R_c$. $R = (4 + 4\beta\cos\eta + \beta^2)^{0.5}R_c$,

To compute I_1, unfortunately, the exclusion distance to the nearest interferer is asymmetric, since the distance to the closest edge of the circle is $R_c + R_g - r$ while the distance to the fur- thest edge is $R_c + R_g + r$. The asymmetric property renders the exact solution generally difficult to obtain except for the special case when $\eta = 4$. To avoid the dependence on the location of the interference field we pursue an upper bound on the interfer- ence power, as illustrated Fig. 2. Specifically, we consider the interference contribution in a ball of radius around the received signal denoted . This result is an upper bound on the aggregate interference due to the fact that less area is excluded from the calculation. Then the receiver is at the center of the reduced interference region, which we call the “small ball” approximation. Under this assumption we write the received power as

$$I_1(r) = \sum_{k\in\Phi\setminus B(R_c+R_g-r)} \frac{G_k R_k P_g}{\ell(R_k)}. \quad (7)$$

We use the same non-singular path-loss model in (6) for the transmit signal power. Other non-singular models and tradeoffs between different models are discussed in [8], [23]. As long as there is a guard region with $R_g > 0$ a singular path-loss model could be employed without changing the results. In our calculations we assume that $R_c + R_g - r > d_0$ to simplify the exposition. To simplify computation in general cases, we also apply the small ball approximation to I_0 to obtain

$$I_0(r) = \frac{G_0 L_0 P_g}{\ell(R_c + R_g - r)} \quad (8)$$

with some abuse of notation.

We will characterize the distribution of the random variable $I(r)$ in terms of its Laplace transform, along the lines of [5]. When the interference marks $G_k L_k$ follow an arbitrary but iden- tical distribution for all the Laplace transform is given by

$$\mathcal{L}_{I(r)}(s) = \mathbb{E}_{I_0(r)}^{k}\left[e^{-sI_0(r)}\right]\mathbb{E}_{I_1(r)}\left[e^{-sI_1(r)}\right]$$

The first term, $\mathbb{E}_{I_0(r)}\left[e^{-sI_0(r)}\right]$, can be directly computed through integral given the distribution of G_0 and L_0. The second term, $\mathbb{E}_{I_1(r)}\left[e^{-sI_1(r)}\right]$, can be computed as

$$\begin{aligned}
\mathcal{L}_{I_1(r)}(s) &= \mathbb{E}_{I_1(r)}\left[e^{-sI_1(r)}\right] \\
&\approx \mathbb{E}_{\Phi,G_k,L_k}\left(e^{-s\sum_{k\in\Phi\setminus B(R_c+R_g-r)}\frac{P_g G_k L_k}{l(R_k)}}\right) \\
&= \mathbb{E}_{\Phi,G_k,L_k}\left(\prod_{k\in\Phi\setminus B(R_c+R_g-r)} e^{-s\frac{P_g G_k L_k}{l(R_k)}}\right) \\
&\overset{(a)}{=} \mathbb{E}_{\Phi}\left(\prod_{k\in\Phi\setminus B(R_c+R_g-r)} \mathbb{E}_{G,L}\left(e^{-s\frac{P_g G L}{l(R_k)}}\right)\right) \\
&\overset{(b)}{=} e^{-2\pi\lambda\int_{R_c+R_g-r}^{\infty}\left(1-\mathbb{E}_{G,L}\left(e^{-sP_g GLC^{-1}v^{-n}}\right)\right)v\,dv} \\
&= e^{\lambda\pi(R_c+R_g-r)^2-\frac{2\pi\lambda(sP_g)^{\frac{2}{n}}}{nC^{2/n}}F_{n,R_c+R_g-r,C}(s)}
\end{aligned} \qquad (9)$$

where (a) follows from the i.i.d. distribution of $G_k L_k$ and its further independence from the point process Φ, (b) follows as- suming that $R_c + R_g - r > d_0$ and using the probability gen- erating functional of the PPP [6], $Z = G_k L_k P_g$, and

$$F_{n,x,C}(s) = \mathbb{E}_z z^{\frac{2}{n}}\left[\Gamma\left(-\frac{2}{n}, sC^{-1}zx^{-n}\right) - \Gamma\left(-\frac{2}{n}\right)\right]$$

In some cases the expectation can be further eval- uated, e.g., when $Z \sim \Gamma[k,\theta]$ then $F_{n,x,C}(s) = \frac{(sC^{-1}x^{-n})^{-\frac{2}{n}-k}n\theta^{-k}}{2+kn}\,{}_2F_1\left(k, k+\frac{2}{n}, 1+k+\frac{2}{n}, -\frac{1}{sC^{-1}x^{-n}\theta}\right)$ $-\theta^{2/n}\mathcal{B}\left(k+\frac{2}{n}, -\frac{2}{n}\right)$ where $\mathcal{B}(x,y)$ is the Beta Euler function and ${}_2F_1$ is the Gauss hypergeometric function.

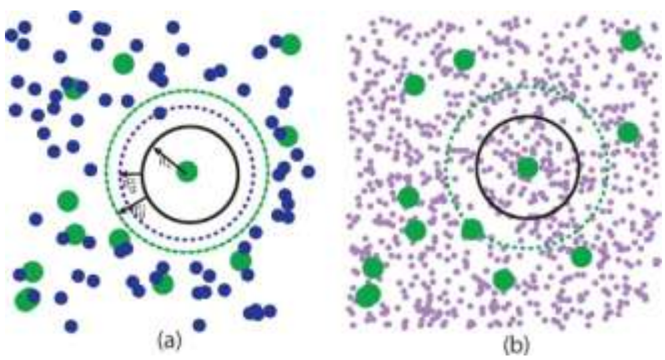


Fig. 3. Interference in a heterogeneous network from multiple kinds of infra- structure. (a) Heterogeneous out-of-cell interference. (b) Homogeneous inter- ference and cross-tier interference from low power nodes.

this case is that there is interference within the cell; effectively the interference contribution is a constant since it is spatially invariant. There may still be a guard radius in this case but it is likely to be small.

A. Heterogeneous Out-of-Cell Interference

Suppose that there are M different kinds of infrastructure de- ployed in a homogeneous way through the network. A more complex model would take non-uniformity or clustering into ac- count [44], but this is beyond the scope of this work. Each inter- ference source can be modeled as a marked PPP with marks cor- responding to the composite fading distribution distributed as $\Gamma[k_m, \theta_m]$ where $m = 1, \ldots, M$. The path-loss function is the same for each process. The transmitting power is given by P_m and the transmitter density by λ_m. Consequently, each process is parameterized by $(k_m, \theta_m, P_m, R_g^{(m)})$.

Since the transmission power of each tier is different, the guard region radius for each tier $R_g^{(m)}$ is different. We assume that the base station associated with the fixed cell belongs to tier 1. Given R_c, the radius of the guard region for tier m, denoted as $R_g^{(m)}$, is

V. HETEROGENEOUS NETWORK INTERFERENCE

In this section we extend the proposed interference model to the case of a heterogeneous network, where the macrocellular network is complemented with other kinds of infrastructure including low-power nodes like small-cells, femtocells, fixed relays, and distributed antennas. We consider two different interference scenarios as illustrated in Fig. 3. The scenario in Fig. 3(a) corresponds to multiple kinds of out-of-cell inter- ference. For example, there might be interference from both high power base stations and low power distributed antenna system transmission points. Since the transmission power of different tiers is heterogeneous, the guard regions associated with different processes may be different. Moreover, the key concepts of the hybrid model, such as the guard region and the dominant interferer are also extended to this case. The scenario in Fig. 3(b) corresponds to cross-tier interference from a second tier of low-power nodes, e.g., small-cells or femtocells or uncoordinated transmission points. The main difference in

$$R_g^{(m)} = (a_m - 1)R_c, \quad (10)$$

where $a_m = 1 + \left(\frac{P_m}{P_1}\right)^{\frac{1}{\alpha}}$. It can be shown that this configuration guarantees that the edge of fixed cell is always covered by its associated base station, neglecting channel fading. Each tier of transmitters forms a homogeneous PPP of density λ_m outside the guard region.

To analyze the performance of a "typical" cell, we now let R_c be proportional to the average radius of the inscribing ball of the typical Voronoi cell in the equivalent stochastic geometry model as we did in the homogeneous case. In that way, the analysis is invariant with the scaling of the base station densities. Given λ_m and P_m, the average radius of the inscribing ball of the typical Voronoi cell can be computed as

$$\frac{1}{2\sqrt{\sum_{m=1}^{M} a_m^2 \lambda_m}}. \quad (11)$$

We denote $\lambda = \sum_{m=1}^{M} a_m^2 \lambda_m$, and let $R_c = \frac{1}{2\sqrt{\sum_{m=1}^{M} a_m^2 \lambda_m}}$ in the following part.

We also extend the concept of the dominant interferer to the heterogeneous model. This case becomes more complex than the homogeneous case: the dominant interferer belongs to tier and locates at the edge of tier m's guard region with probability $\frac{a_m^2 \lambda_m}{\lambda}$.

In the M-tier network, the total interference is made up of
1 : from the dominant interferer; I_m $(1 < m \leq M)$ from the base stations outside the guard region of tier , which is shot-field noise. Furthermore, I_m $(1 \leq m \leq M)$ are inde- pendent. The equation for the total interference under the small ball approximation is given by

$$J_{\text{tot}}(r) = \sum_{m=1}^{M} \sum_{k \in \Phi_m \setminus B(R_c + R_g^{(m)} - r)} \frac{G_k^{(m)} L_k^{(m)} P_m}{\ell\left(R_k^{(m)}\right)} + I_0.$$

From the independence of the M PPPs, the Laplace transform of the heterogeneous out-of-cell interference is

$$\mathcal{L}_J(s) = \prod_{m=0}^{M} \mathcal{L}_{I_m(r)}(s),$$

with $\mathcal{L}_{I_m(r)}(s)$ the Laplace transform of $I_m(r)$.

By (9), it is easy to find that the Laplace transform of $I_m (1 < m \leq M)$ is given by

$$\exp\left[\lambda\pi\left(R_c + R_g^{(m)} - r\right)^2 - \frac{2\pi\lambda(sP_m)^{\frac{2}{n}}}{nC^{2/n}} F_{n, R_c + R_g^{(m)} - r, C}(s)\right]$$

B. Cross-Tier Interference

Certain kinds of infrastructure, like femtocells, are not associ- ated with the macro base station. Rather they form another tier of nodes and create cross-tier interference. We propose to use the same stochastic geometry framework to model the interference with a main difference in how the notion of guard zone is de- fined. Let B_{cross} denote an exclusion region around the desired receiver with radius R_{cross}. This region might be quite small, just a few meters radius, and is designed to avoid the case where the low-power node is co-located with the target receiver. Let us suppose that the cross-tier interference is associated with a PPP given by Φ_{cross}, the transmitting power is given by P_{cross}, and the transmitter density by λ_{cross}. The received interference power is given by

$$I_{\text{cross}} = \sum_{k \in \Phi_{\text{cross}} \setminus B_{\text{cross}}} \frac{G_k L_k P_{\text{cross}}}{\ell(R_k)},$$

which does not depend on r due to the shift invariance prop- erty of the PPP. Consequently cross-tier interference creates a constant interference that is independent of the mobile receiver location.

VI. APPROXIMATING THE INTERFERENCE DISTRIBUTION USING THE GAMMA DISTRIBUTION

The expressions in Section IV and V provide a characteriza- tion of the distribution of the interference terms in the proposed hybrid approach. Unfortunately, the distributions are character- ized in terms of the Laplace transform and further simplifica- tions exist only for special choices of the parameters and mark distribution. Consequently, in this section, we pursue an approx- imation of the distribution of the interference using the Gamma distribution. First we review the calculations for the Gamma ap- proximation of the interference terms with a single and multiple interferers. Then we use these calculations to parameterize the Gamma approximation of the interference.

A. Moments of the Interference Process

The moments for interference distribution can be computed along the lines of ([8], Equations 2.19 and 2.21). We only need the first two moments for the Gamma approximation. Let $G_k L_k P_g$ denote the random variable corresponding to the mark distribution for the case of a homogeneous interference source.

For general cases, we use the approximation expression as in (7) and (8) to derive the following proposition.

Proposition 7 (Moments of Homogeneous Interference): The mean and variance of the interference in the case of

$$\mathbb{E}I(r) \approx \quad \frac{\mathbb{E}Z}{C(R_c + R_g - r)^n} \left[\frac{2\pi\lambda(R_c + R_g - r)^2}{(n-2)} + 1\right],$$

$$\frac{1}{C^2(R_c+R_g-r)^{2n}} \left[\frac{\pi\lambda\mathbb{E}Z(R_c+R_g-r)^2}{(n-1)} + \mathbb{V}Z\right]$$

homoge- neous interference are

Z

$$\mathbb{V}I(r) \approx$$

In the case of heterogeneous interference, we propose to use the moments of the sum to approximate using the equivalent mark distribution. We summarize the key results in the fol- lowing proposition.

Proposition 8 (Moments of Heterogeneous Interference): The mean and variance of the interference in the case of het- erogeneous interference are

$$\mathbb{E}I_0(\beta R_c) = \quad \frac{1}{\lambda C R_c^4} \sum_{m=1}^{M} \frac{\lambda_m \mathbb{E}Z^{(m)} a_m^2 (a_m^2 + \beta^2)}{(a_m^2 - \beta^2)^3},$$

$$\mathbb{E}I_0^2(\beta R_c) = \frac{1}{\lambda C^2 R_c^8} \sum_{m=1}^{M} \lambda_m \mathbb{E}\left(Z^{(m)}\right)^2 \times \frac{a_m^2 (a_m^2 + \beta^2)(a_m^4 + 8a_m^2\beta^2 + \beta^4)}{(a_m^2 - \beta^2)^7},$$

$$\mathbb{E}I_m(\beta R_c) = \frac{2\pi\lambda_m \mathbb{E}Z^{(m)} a_m^2}{2C\ R_c(\beta^2 - a_m^2)^2},$$

$$\mathbb{V}I_m(\beta R_c) = \frac{2\pi\lambda_m \mathbb{E}\left(Z^{(m)}\right)^2 a_m^2\ (a_m^4 + 6a_m^2\beta^2 + 3\beta^4)}{6C^2 R_c^6\ (a_m^2 - \beta^2)^6}.$$

where .

$Z_c^{(m)}\ \ G_k^{(m)} \beta^2 L_k^{(m)} P_m$

When , we can calculate the moments of the interfer- ence in closed form without the small ball approximation.

Proposition 9 (Closed Form of Moments No Small Ball): When , the exact moments of heterogeneous interference can be computed through

$n = 4$

$$\mathbb{E}I_0(\beta R_c) = \quad \frac{1}{\lambda C R_c^4} \sum_{m=1}^{M} \frac{\lambda_m \mathbb{E}Z^{(m)} a_m^2\ \left(a_m^2 + \beta^2\right)}{\left(a_m^2 - \beta^2\right)^3},$$

$$\mathbb{E}I_0^2(\beta R_c) = \quad \frac{1}{\lambda C^2 R_c^8} \sum_{m=1}^{M} \lambda_m \mathbb{E}\left(Z^{(m)}\right)^2 \times \frac{a_m^2\ \left(a_m^2 + \beta^2\right)\ \left(a_m^4 + 8a_m^2\beta^2 + \beta^4\right)}{\left(a_m^2 - \beta^2\right)^7},$$

$$\mathbb{E}I_m(\beta R_c) = \quad \frac{2\pi\lambda_m \mathbb{E}Z^{(m)} a_m^2}{2C\ R_c\ \left(\beta^2 - a_m^2\right)^2},$$

$$\mathbb{V}I_m(\beta R_c) = \quad \frac{2\pi\lambda_m \mathbb{E}\left(Z^{(m)}\right)^2 a_m^2\ \ \left(a_m^4 + 6a_m^2\beta^2 + 3\beta^4\right)}{6C^2\ R_c^6\ \left(a_m^2 - \beta^2\right)^6}.$$

Lastly, an additional term could be included accounting for the cross-tier interference I_{cross} in a straightforward way. Fur- ther modifications are possible. For example, having different numbers of active users in an interfering cell with multiuser MIMO could be included by having a sum of marked point pro- cesses each with a different fading distribution corresponding to different choices of active users. Note that we can incorpo- rate cross-tier interference I_{cross} into Proposition 8 by replacing $R_c + R_g - r$ with R_{cross} and selecting the other parameters as appropriate.

B. Gamma Approximation of the Interference

While the Laplace transform of the interference power com- pletely characterizes the distribution, it is often challenging to provide simple or closed-form solutions for a variety of sce- narios of interest. Consequently, we pursue an approximation of the interference term that will yield simpler expressions targeted at the proposed hybrid setting using the Gamma distribution.

Proposition 10 (Gamma Approximation of Interference): The random variable with the same mean and variance as has

(12) $k_i =$ $$\theta_i = \frac{\mathbb{V}I}{\mathbb{E}I}.$$

VII. CHARACTERIZATION OF THE RANDOM SIR

In this section we characterize the system performance as a function of the random quantity the signal to interference ratio (SIR) given by

$$\text{SIR}(r) = \frac{S(r)}{J(r)}.$$

The SIR is a useful quantification for performance analysis in cellular systems because performance at the cell edge is usually interference limited; we consider additive noise and signal to interference plus noise ratio (SINR) in the simulations. Because simplified expressions including the exact distribution calculated in Sections IV–V are only available in special cases, we use the Gamma approximation described in Section VI. Tomake the results concrete, we assume that $S(r) = S_r$ where $S_r \sim \Gamma[k_s, \theta_s]$. Note that in the distant-dependent path-loss and transmit power are absorbed into the θ_s term for ease of presentation.

We use the $\text{SIR}(r)$ to evaluate two metrics of performance: the success probability defined as $\mathbb{P}(\text{SIR}(r) > T)$ where is some threshold and the ergodic achievable rate given by

$$\tau(r) := \mathbb{E}\ln(1 + \text{SIR}(r)).$$

The success probability is one minus the outage probability and gives a measure of diversity performance and severity of the fading. The achievable rate gives the average rate that can be achieved at assuming that the interference is Gaussian.

r

A. Success Probability

We first provide the success probability without proceeding to a Gamma approximation for the out-of-cell interference and signal distribution given by $S(r) = S_r$ where $S_r \sim \Gamma[k_s, \theta_s]$. Using the results from [41], the success probability at location r is given by

(13) $$\mathbb{P}(\text{SIR}(r) > T) = \sum_{j=0}^{k_s-1} \frac{1}{j!} \left(-\frac{\ell(r)T}{P_s\theta_s}\right)^j \frac{\mathrm{d}^j \mathcal{L}_I(s)}{\mathrm{d}s^j}\bigg|_{s=\frac{\ell(r)T}{P_s\theta_s}}$$

where in the case of heterogeneous interference, $\mathcal{L}_I(s)$ is given by (9). In the special case of $k_s = 1$ the success probability can be simplified as

$$\mathbb{P}(\text{SIR}(r) > T) = \mathcal{L}_I(\ell(r)T/(P_s\theta_s)). \quad (14)$$

Evidently, the closed-form expression (13) is involved and no useful insight can be obtained. This motivates the use of Gamma approximation for the aggregate interference as a means to pro- vide simpler and useful expressions.

We evaluate below the success probability for the case of Gamma distributed interference, effectively $J(r)$. In the case of homogeneous interference, the expression for and θ_i is found (12). In the case of heterogeneous interference, the expressions for k_i and θ_i are given by (12), replacing $I(r)$ by $J(r)$. The result is summarized in the following proposition.

Proposition 11 (Success Prob. for Gamma Interference): The success probability when the signal distribution is where and the interference distribution is I_r where is given by $S_r \sim \Gamma[k_s, \theta_s]$ $I_r \sim \Gamma[k_i, \theta_i]$ $S(r) = S_r$ $J(r) =$

$$\mathsf{P}(\mathrm{SIR}(r) > T) = \frac{\Gamma(k_s + k_i)}{\Gamma(k_s)} \left(\frac{\theta_s}{T\theta_i}\right)^{k_i} \times {}_2\tilde{F}_1\left(k_i, k_s + k_i, 1 + k_i, -\frac{\theta_s}{T\theta_i}\right),$$

where ${}_2\tilde{F}_1$ is a regularized hypergeometric function.

Proof: First we rewrite the probability to separate the signal and interference terms as $\mathsf{P}(\mathrm{SIR}(r) > T)$ $\mathsf{P}(S_r > TI_r)$. Now conditioning on the interference and eval- uating the expectation

$$\mathsf{P}(\mathrm{SIR}(r) > T) = \mathsf{E}(F_S(TI_r)) = \frac{\Gamma(k_s + k_i)}{\Gamma(k_s)} \left(\frac{\theta_s}{T\theta_i}\right)^{k_i} {}_2\tilde{F}_1\left(k_i, k_s + k_i, 1 + k_i, -\frac{\theta_s}{T\theta_i}\right),$$

where $F_S(\cdot)$ is the cumulative distribution function of S_r and

[46], [47] while for larger values of the asymptotic expres- $|z|$

the second equality follows from evaluating the expectation. ■

We can obtain a more exact expression in the case where each interferer is separately assumed to have a Gamma distribution.

Proposition 12 (Heterogeneous Success Probability): The success probability when the signal distribution is where and the interference distribution is $S(r) = S_r$ where $I_r^{(m)}$ for is given by $J(r) =$ $\sim \Gamma[k_m, \theta_m]$ $m =$ $S_r \sim \Gamma[k_s, \theta_s]$ $J_r^{(1)} + J_r^{(2)} + \ldots + J_r^{(M)}$ $1, 2, \ldots, M$

$$\mathsf{P}(\mathrm{SIR}(r) > T) = C \sum_{n=0}^{\infty} c_n \frac{\Gamma(k_s + \rho + n)}{\Gamma(k_s)} \left(\frac{\theta_s}{T\theta_{\min}}\right)^{n+\rho} \times {}_2\tilde{F}_1\left(n + \rho, k_s + n + \rho; 1 + n + \rho; -\frac{\theta_s}{T\theta_{\min}}\right),$$

where the parameters C, ρ, θ_{min} are found via Theorem 4.

Proof: First we rewrite the probability to separate the signal and interference terms, and then conditioning on the interference, we evaluate the expectation, i.e.,

$$\mathsf{P}(\mathrm{SIR}(r) > T) = \mathsf{P}\left(S_r > T \sum_m I_r^{(m)}\right) = \mathsf{E}\left(F_S\left(T \sum_m I_r^{(m)}\right)\right).$$

From the results of Theorem 4, the distribution of $\sum_m I_r^{(m)}$ can be expressed as

sion can be employed [48]. Depending on which numerical package is employed, it may be faster to compute the SIR using Monte Carlo techniques based on and . $\Gamma[k_s, \theta_s]$ $\Gamma[k_i, \theta_i]$

B. Average Rate

The average rate is useful for computing average rates at the cell edge and the area spectral efficiency, where the average rate at is integrated over the radius of the cell. Our calcula- tions are be based on the observation that . From Lemma 2, has a com- $\mathsf{E}\ln(S_r + I_r) - \mathsf{E}\ln(I_r)$ r

putable solution but requires dealing with a sum of Gamma random variables. To solve this problem, we can use the result in (2) to find an expression for the expected log of the sum of Gamma random variables. The $\mathsf{E}\ln(S_r + I_r)$

resulting expression is summarized in the following proposition.

Proposition 13: Suppose that $\{X_m\}_{m=1}^{M}$ are independent distributed random variables. Then $\Gamma[k_m, \theta_m]$

$$\mathbb{E}\ln\left(\sum_{m=1}^{M} X_m\right) = \psi(\rho) + \ln(\theta_{min}) + C\sum_{n=0}^{\infty} c_n \sum_{m=0}^{n-1} \frac{1}{\rho+i}.$$

Proof: See ([37], Proposition 5).

Using this result, the average rate with homogeneous inter- ference follows.

Proposition 14 (Homogeneous Average Rate): The average rate with homogeneous interference in the absence of noise is

$$\tau(r) = \mathbb{E}\ln(1+\mathrm{SIR}(r))$$

=

$$f_Y(y) = C\sum_{n=0}^{\infty} \frac{c_n}{\Gamma(\rho+n)\theta_{min}^{\rho+n}} y^{\rho+n-1} e^{-y/\theta_{min}}$$

$$C\sum_{n=0}^{\infty} c_n \ln(\theta_{min}) + C\sum_{n=0}^{\infty} c_n \psi(\rho+n) - \psi(k_i) - \ln(\theta_i),$$

= = ρ $\sum_{m=1}^{M} k_i^{(m)}$, C
is found by recursion from

where $\theta_{min} := \min_m \theta_i^{(m)}$, $\prod_{m=1}^{M}(\theta_{min}/\theta_i^{(m)})^{k_i^{(m)}}$, and c_m

$$c_{n+1} = \frac{1}{n+1}\sum_{p=1}^{n+1} p\gamma_p c_{n+1-p}$$

where $\theta_{min} := \min\{\theta_s, \theta_i\}$, $(\theta_{min}/\theta_s)^{k_s}(\theta_{min}/\theta_i)^{k_i}$ c_n

(3).

$\rho = k_s + k_i$, $C =$
, and is found by recursion in

γ_p with $\sum_{m=1}^{M}$. Now recognize $\theta_{min}/\theta_i^{(m)})^n$
that (2) is essentially a mixture of $\Gamma[\rho+n, \theta_{min}]$ random vari- ables with weights c_n. Now we can use the results of (11) to establish that

$$\mathsf{P}(\mathrm{SIR}(r) > T) = C \sum_{n=0}^{\infty} c_n \frac{\Gamma(k_s+\rho+n)}{\Gamma(k_s)} \left(\frac{\theta_s}{T\theta_{\min}}\right)^{n+\rho} \times {}_2\tilde{F}_1\left(n+\rho, k_s+n+\rho; 1+n+\rho; -\frac{\theta_s}{T\theta_{\min}}\right).$$

Proof: Follows from the application of Proposition 13 to the case of the sum of two Gamma random variables $\Gamma[k_i, \theta_i]$ and $\Gamma[k_s, \theta_s]$, and using the result in Lemma 2.

The average rate with heterogeneous interference can be computed along the same lines as Proposition 14.

Proposition 15 (Heterogeneous Average Rate): The average rate with homogeneous interference in the absence of noise is

$$\tau(r) = C \sum_{n=0}^{\infty} c_n \ln\left(\theta_{min}^{(si)}\right) + C \sum_{n=0}^{\infty} c_n \psi\left(\rho^{(si)} + n\right) - D \sum_{n=0}^{\infty} d_n \ln\left(\theta_{min}^{(i)}\right) - D \sum_{n=0}^{\infty} d_n \psi\left(\rho^{(i)} + n\right),$$

■

While the result in Proposition 12 is more exact, it requires truncating the infinite series. In related work [37] we found that in most cases the approximation outperforms the truncated ex- pression in (2) unless a large number of terms are kept, and therefore find Proposition 11 is sufficient.

The regularized hypergeometric function ${}_2\tilde{F}_1(a, b, c, z)$ is

where $\theta_{min}^{(si)} := \min\{\theta_s, \theta_i^{(1)}, \theta_i^{(2)}, \ldots, \theta_i^{(M)}\}$, $\rho^{(si)} = k_s + k_i^{(1)} + \cdots + k_i^{(M)}$, $\theta_{min}^{(i)} := \min\{\theta_i^{(1)}, \theta_i^{(2)}, \ldots, \theta_i^{(M)}\}$, $C = (\theta_{min}/\theta_s)^{k_s} \prod_{m=1}^{M} (\theta_{min}^{(si)}/\theta_i^{(m)})^{k_i^{(m)}}$, $\rho^{(i)} = k_i^{(1)} + \cdots + k_i^{(M)}$, $D = \prod_{m=1}^{M} (\theta_{min}^{(i)}/\theta_i^{(m)})^{k_i^{(m)}}$, and c_m and d_m are found by recursions like in (3).

available in many numerical packages or can be computed from unregularized hypergeometric function by recognizing that ${}_2\tilde{F}_1(a, b, c, z) = \frac{{}_2F_1(a,b,c,z)}{\Gamma(c)}$. For small values of $|z|$ it is convenient to compute it using the truncated series definition

Proof: The first pair of terms result from application of Proposition 13 to the term $\mathsf{E} \ln S_r + I_r^{(1)} + I_r^{(2)} + \cdots + I_r^{(M)}$,
while the second term results from application of Proposition 13 to the term $\mathsf{E} \ln I_r^{(1)} + I_r^{(2)} \cdots I_r^{(M)}$.

If the number of terms is too large for practical computation, an alternative is to approximate the sum of the signal plus in- terference term as another Gamma random variable using the moment matching approach.

Proposition 16 (Homogeneous Average Rate—Gamma): The average achievable rate without noise with homogeneous inter- ference is approximately

$$\tau(r) \approx \psi\left(\frac{(k_s\theta_s + k_z\theta_z)^2}{k_s\theta_s^2 + k_z\theta_z^2}\right) + \ln\left(\frac{k_s\theta_s^2 + k_z\theta_z^2}{k_s\theta_s + k_z\theta_z}\right) - \psi(k_i) - \ln(\theta_i).$$

Proof: To find $\mathbb{E}\ln(S_r + I_r)$ we apply Lemma 5 and ap- proximate the sum of Gamma random variables with another Gamma distribution with the same mean and variance. Note that

$$\mathbb{E}\left[S_r + I_r\right] = k_s\theta_s + k_i\theta_i$$
$$\mathbb{V}\left[S_r + I_r\right] = k_s\theta_s^2 + k_i\theta_i^2.$$

Then a Gamma random variable with the same mean and variance has

$$k_z = \frac{(k_s\theta_s + k_i\theta_i)^2}{k_s\theta_s^2 + k_i\theta_i^2}\theta_z = \frac{k_s\theta_s^2 + k_i\theta_i^2}{k_s\theta_s + k_i\theta_i}. \quad Z \sim \Gamma[k_z, \theta_z]$$

Now we can approximate the term

$$\mathbb{E}\ln(S_r + I_r) \approx \psi\left(\frac{(k_s\theta_s + k_i\theta_i)^2}{k_s\theta_s^2 + k_i\theta_i^2}\right) + \ln\left(\frac{k_s\theta_s^2 + k_i\theta_i^2}{k_s\theta_s + k_i\theta_i}\right)$$

where the last part follows from Lemma 2. ■

The result can be extended to the case of heterogeneous inter- ference by modifying the computation of the mean and variance terms.

VIII. SIMULATIONS

In this section we consider a system model of the form in Fig. 2 with a single antenna base station and a Poisson field of base station interferers. The density of interferers λ_1 equals $\frac{1}{16\times 300^2}$ transmitters per m^2. We evaluate performance in a fixed cell of radius $R_c = \frac{1}{\sqrt{\pi\lambda_1}} = 300$ m which is the radius of the typical cell to make an equivalent comparison with the sto- chastic geometry model. In the homogeneous case, the radius of the guard region R_g is 300 m. The macro base station has 40 W transmit power and Rayleigh fading is assumed. The path-loss model of (6) is used with $\alpha = 4$, $d_0 = 35$ m, and $G = 33.88$. Log-normal shadowing is assumed with 6 dB variance. Monte Carlo simulations are performed by simulating base station locations over a square of dimension $15R_c \times 15R_c$. The mutual information and the outage probability are estimated from their sample averages over 200 small scale fading realizations, and 500 different interferer positions. Performance is evaluated as a function of $r = \beta R_c$, the distance from the center of the fixed cell.

First, we validate two of the key concepts used in the pro- posed model in this paper: the use of guard region and of the dominant interferer. In Fig. 4 we compare the performance be- tween the proposed hybrid approach and a PPP layout of base stations as in the stochastic geometry model. Our proposed hy- brid approach matches the stochastic geometry model well at

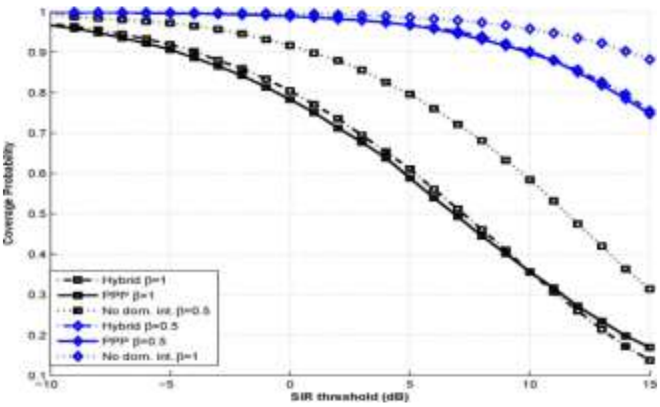


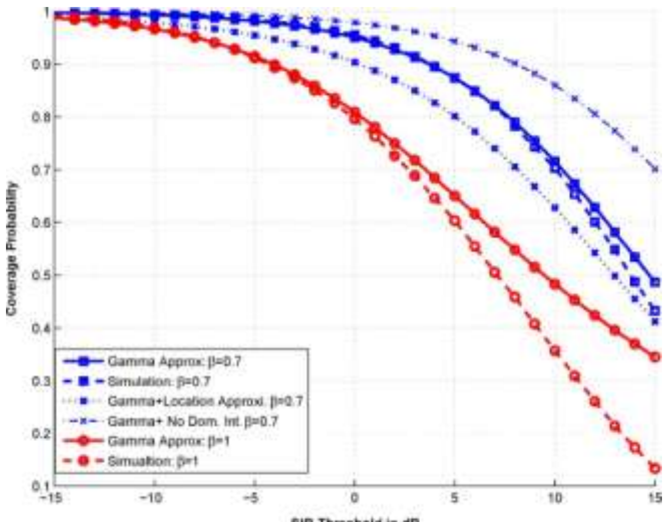
Fig. 4. Coverage probability as a function of $r = \beta R_c$ the distance from the cell center. Our proposed hybrid model matches the stochastic geometry model where base stations are distributed as a PPP. Further, we show that the dominant interferer is essential; without a dominant interferer, the error becomes large.

all locations in terms of SIR distribution. Without including the dominant interferer at the cell edge, however, the result is too optimistic.

Now we consider the Gamma approximation for the inter- ference field. We claim that the Gamma distribution is a good approximation of the distribution strength of the interference power. Also, we show that the approximation that the inter- ference power for a user at radius r from the cell center is upper bounded by considering interferers outside a ball of ra- dius $R_c + R_g - r$ provides a relatively tight bound in terms of SIR distribution.

Now we compare the Gamma approximation in terms of the average rate and the success probability in a case with hetero- geneous interference. We consider a setup with a single antenna base station and three sources of interference. One source of interference is from other base stations, which has power and density as we already described. The second source of inter- ference is a multiuser MIMO distributed antenna system (DAS) with four single antenna transmission points, a single active user per cell. Each DAS antenna has 20 W transmission power. The DAS system is also modeled using a PPP but with density $4\lambda_1$. The distributions of the effective channels are derived from [37]. Note that essentially we assume that every cell also has a DAS system; in reality some cells would have DAS and some would not, which can be taken into account by modifying the density. The third source of interference is from cross-tier interference from a single tier of femtocells. The femto interferers have 20 dBm transmit power and a guard region of 5 m. Since this is cross-tier interference, the femtocells may be deployed in the cell and thus they create a constant level of interference power corresponding to a minimum distance of 5 m. An indoor-out- door penetration loss of 10 dB is assumed.

An important point is that, compared with the homogeneous case, the size (effective coverage area) of the fixed cell shrinks in the heterogeneous networks due to the increase of the overall in- terferer density. The reason is that we calculate the radius of the



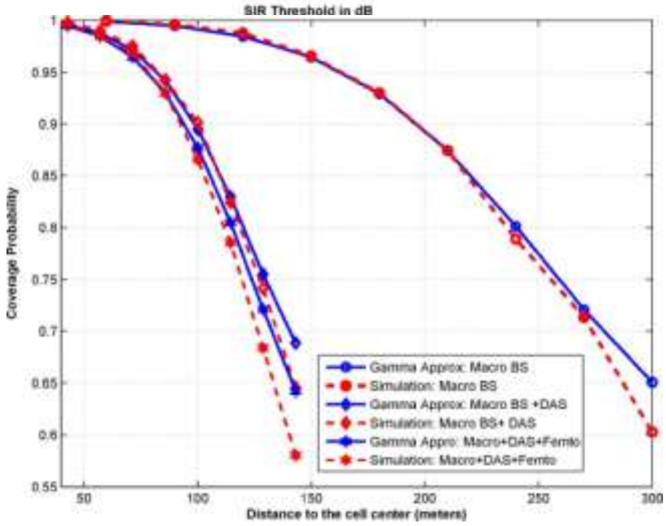


Fig. 5. Coverage probability for a receiver at a distance $0.7R_c$ from the center. The curves show that Gamma distribution is a good approximation of the inter- ference power. Moreover, the approximation to avoid asymmetric receiver lo- cations provides a relatively tight bound in terms of coverage probability. The error becomes more significant at the cell edge when for high SIR, which is not a huge problem since the likelihood of large SIR at the cell edge is small.

Fig. 6. Comparison of the ergodic rates with and without the Gamma approxi- mation with a tier of low-power femto nodes that create cross-tier interference. In the heterogeneous case, R_c is 143.20 meters, which is smaller than in the homogeneous case. We see that the Gamma approximation provides low error except towards the cell edge for the case. Cross-tier interference provides a con- stant offset of the entire curve of no cross-tier interference case. Thanks to the 10 dB indoor-outdoor penetration loss, the offset is relatively small. The offset, notably, in both Gamma approximation and simulations is consistent, meaning it is still possible to make relative comparisons between the Gamma approxi- mation curves.

heterogeneous cell and the size of the guard region for each tier from (10) and (11). Consequently, the coverage area is smaller when the network is more dense, as expected from intuition.

The results of the comparison are displayed in Fig. 6 for av- erage capacity and in Fig. 7 for outage. In terms of average rate, the main conclusion is that the Gamma approximation provides good performance for both homogeneous and heterogeneous in- terference. There is more significant error at the cell edge in both cases. In terms of success probability, similar results are

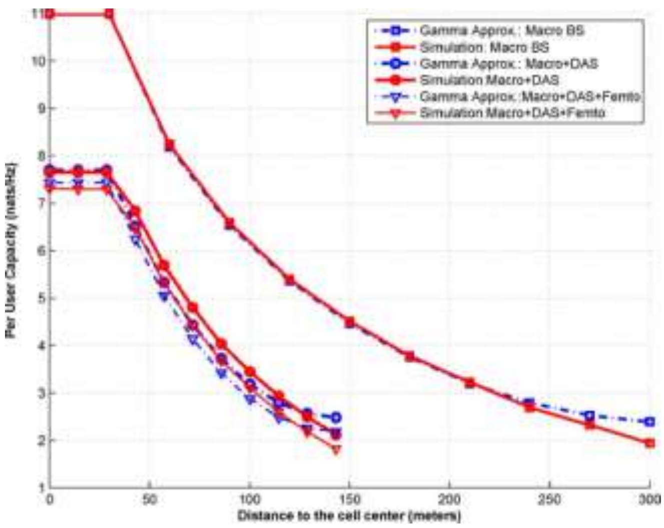


Fig. 7. Comparison of the success probability between the Gamma approxi- mation and Monte Carlo simulations for a target SIR of 5 dB. We see that the Gamma approximation overall provides reasonable performance with a slight error at the cell edge.

observed. The fit with both homogeneous and heterogeneous interference is good, except for locations at the cell edge. While there is approximation error, we still believe that the fit is good enough to justify the viability of the Gamma distribution.

IX. CONCLUSION

In this paper, we proposed a hybrid model for determining the impact of interference in cellular systems. The key idea was to develop a model for the composite interference distribution out- side a fixed cell, as a function of the user position from the cell center. From a numerical perspective, our approach simplifies the simulation study of cellular systems by replacing the sum in- terference term with an equivalent interference random variable. Then, functions like average rate and success probability can be computed through numerical integration rather than Monte Carlo simulation. Unfortunately, the numerical integrals may still require a fair amount of computational power to compute. Consequently, we proposed to approximate the interference distribution by moment matching with the Gamma distribution. We showed how the Gamma distribution could be used to obtain relatively simple expressions for the success probability and er- godic rate, which simplify further when the sum interference power is approximated directly as a Gamma random variable. Simulations showed that with the introduction of guard region and a dominant interferer, a reasonable fit between the stochastic geometry model and the proposed model was achieved, with a small error at the cell edge when the Gamma approximation was employed. Future work is needed to better characterize the ap- proximations, e.g., by developing expressions for bounding the error terms, and also incorporating more complex propagation models like those proposed in [49].

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